CRMT Team Round

CNCM Administration

October 26th, 2019

1. What is $\left(2-\frac{0}{1}\right)\left(2-\frac{1}{2}\right)\left(2-\frac{2}{3}\right)\left(2-\frac{3}{4}\right)\left(2-\frac{4}{5}\right)\left(2-\frac{5}{6}\right)\left(2-\frac{6}{7}\right)\left(2-\frac{7}{8}\right)\left(2-\frac{8}{9}\right)$?

2. Define the sequence a_n by $a_1 = 1$ and $a_n = a_{n-1} + n$ for positive integers $n > 1$. Find the positive integer n such that $a_n = 35n$.

3. A circle with radius 1 is inscribed inside a right triangle. Find the minimum area of the triangle in simplest radical form.

4. Let $f(x) = \frac{\ln\left(\frac{1+x}{1-x}\right)}{2}$. Compute

$$\lim_{x\to\infty} f^{-1}(x).$$

5. Find the last 3 digits of the base 10 number 5^{2019} when converted to base 6.

6. Suppose there are 60 lily pads arranged in a circle and a frog is on one of them. Every second, the frog moves to one of the two adjacent lily pads with equal probability. After how many seconds is the frog expected to return to the initial lily pad?

7. Let f be the fifth-degree polynomial such that $f(1) = 0$, $f(2) = 60$, $f(3) = 60$, $f(4) = 60$, $f(5) = 60$, $f(6) = 120$, and $f(7) = 480$. Compute $f(8) + f(9)$.

8. Define the vertices of an infinite-sided polygon P inscribed in the unit circle (centered at the origin) as follows: $V_1 = (0, -1)$, $V_2 = (-1, 0)$, and V_N is the midpoint of the minor arc $\overset{\frown}{V_{N-1}Z}$ for integers $N > 2$, where Z is the point $(0, 1)$. Measured in units and square units respectively, find the ratio between the perimeter of the polygon and its area.

9. A deck of the first 100 positive integers is randomly shuffled. Find the expected number of draws it takes to get a prime number if there is no replacement.

10. A cube has side length 12. Let F be a face of the cube and let $ABCD$ be the square corresponding to face F on the cube. Pick a point P such that $\overline{PA} = \overline{PB} = \overline{PC} = \overline{PD} = \sqrt{136}$ and P lies outside the cube. Let S_1 be the inscribed sphere of the cube and let S_2 be the inscribed sphere of square pyramid $PABCD$. There exists a cone C such that both S_1 and S_2 are tangent to the lateral surface of the cone and S_1 is tangent to the base of the cone. Compute the volume of cone C.