CENTRAL NC
MATH GROUP

Logarithms

By: Minseok Eli Park

Introduction to Logarithms

- Logarithms, called logs for short, are convenient ways to denote powers of numbers.

- Here is the mathematical definition.

If $$b^x = y$$

Then $$\log_b y = x$$

- Where b, y are both positive.

- If the base (b) is not present, it is assumed that b = 10.
- If instead of log it says ln, it is assumed that b=e.

- Notice that the parent exponential function and the parent log function with the same base are inverses.

Exercises

- Evaluate the following.
- 1. $$\log_8 512$$
- 2. $$\log_{\sqrt{3}} 3$$
- 3. $$\log 1000$$
- 4. $$\ln e^2$$

Properties

- The following properties will allow us to simplify relations of logs and start to use them creatively.

$$1.\ \log_a b^n = n \log_a b$$

$$2.\ \log_a b + \log_a c = \log_a bc$$

$$3.\ \log_a b - log_a c = \log_a \frac{b}{c}$$

$$4.\ (\log_a b)(\log_c d) = (\log_a d)(\log_c b)$$

$$5.\ \frac{\log_a b}{\log_a c} = \log_c b$$

$$6.\ \log_{a^n} b^n = \log_a b$$

Properties

- Many of these relations are direct, by converting them to its exponential definition, and applying exponential properties.
- But do not rely on this tactic to solve logarithm problems, you will likely need these properties.

- A few comments:
- #1 is demonstrated by exercise #4 on slide 3. Observe how you can generalize that problem into the property.
- #2 and #3 are essentially the same, as multiplication/division maneuvers into addition/subtraction in the language of logarithms.
- #4 when b = c, $log_b c$ cancels to 1, making this a nice property.
- #5 is important if you have an ancient calculator that only allows logs base 10. Just use a=10 and this property to find logs of other bases.
- #6 is shown in exercise #1 on slide 3. It is easy to generalize this problem into this property.

Example 1

- Find all x such that the following equation is true.

$$\log_{69}(x+3) + \log_{69}(x+23) = 1$$

- First, we use property #2 to convert the sum of 2 logs with the same base into 1.

$$\log_{69}(x+3)(x+23) = 1$$

- Now using our definition of logs, we convert this into exponential form, which is:

$$(x+3)(x+23) = 69^1$$

Example 1

- Expanding on the LHS and subtracting both sides by the RHS, we get the simple quadratic that factors into the following:

$$x(x+26)=0$$

- It may seem reasonable to think that x = -26, 0 and finish, but this is not the issue.
- You know those things that occur in square root problems? Yes, those pesky ***extraneous solutions***.
- We plug both solutions back to realize that we cannot have negatives inside of a logarithms (refer back to the definition page).
- The only solution is then x = 0.

Example 2

- Evaluate the following product

$$(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$$

- We abuse property #4 and its corollary. Take the first two terms. Simplifying makes this become the following.

$$(\log_2 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$$

- We do this again on next two terms.

$$(\log_2 5)(\log_5 6)(\log_6 7)(\log_7 8)$$

- Doing this continuously simplifies to $log_2 8$ which is just 3.
- This is the well known chain rule of logs.

Example 3

- (AHSME) Find all x such that the following equation is true.

$$x^{\log x} = \frac{x^3}{100}$$

- This one is a bit of challenge, because it takes a very important step that may seem foreign to students relatively new to logs.

- First, we take the log of both sides (base 10).
- This step is not completely arbitrary. We did this, because we know can simplify many parts of this equation to get log x everywhere, which we may eventually make a substitution for.
- Taking log on both sides brings us to the following equation.

$$\log x^{\log x} = \log \frac{x^3}{100}$$

Example 3

- First, we use property #3 on the RHS.

$$\log x^{\log x} = \log x^3 - 2$$

- Now using property #1 on both sides, we have the following.

$$(\log x)(\log x) = 3\log x - 2$$

- Gosh, my hypothesis was correct. We have log x everywhere. The substitution log x = a seems convenient.

$$a^2 = 3a - 2$$

$$a^2 - 3a + 2 = 0$$

Example 3

- Solving this simple quadratic gives a = 1, 2. Substituting this back into our definition of a, we have the following:

$$\log x = 1, 2$$

$$x = 10^1, 10^2$$

- For completion, we plug these solutions back into the original equation. No problems arise, so our final answer is x = 10, 100.

Challenges

- Here are some challenges not from boomer contests like AHSME.

- 2020 AMC 12A #10 https://artofproblemsolving.com/wiki/index.php/2020_AMC_12A_Problems/Problem_10
- 2007 AIME I #7 https://artofproblemsolving.com/wiki/index.php/2007_AIME_I_Problems/Problem_7
- 2020 February HMMT Algebra and Number Theory #3 https://hmmt-archive.s3.amazonaws.com/tournaments/2020/feb/algnt/problems.pdf
- 2010 AIME I #14 https://artofproblemsolving.com/wiki/index.php/2010_AIME_I_Problems/Problem_14

Thanks for Watching!

- Credit to AoPS Volume 2 for list of properties.
- Thanks to Rishabh Bedidha and Harry Chen for collection of challenge problems.