Roots of Unity POTD Solutions

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Week of 11/10

Monday (ARML). Find the sum of all integer values of x between 0 and 90 inclusive so that $(\cos(x^\circ) + i\sin(x^\circ))^{75}$ is a real number.

Solution. By Euler's identity, we just need

$$\cos(75x) + i\sin(75x)$$

to be a real number. That means, the argument, which is $75x$, must be a multiple of 180. That is,

$$\frac{75}{180}x = \frac{5}{12}x$$

must be an integer. Hence, x just needs to be a multiple of 12. The sum we are looking for is thus

$$12\cdot 0 + 12\cdot 1 + \cdots + 12\cdot 7 = 12\cdot\frac{7\cdot 8}{2} = 12\cdot 28 = \boxed{336}$$

Tuesday (Moldovan TST). Let S be the set of all positive integers that have 100 digits, are divisible by 3, and only contain digits in the set $\{3, 5, 7, 9\}$. Find the remainder when the number of elements of S is divided by 29.

Solution. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Hence, we simply need to choose from the set $\{3, 5, 7, 9\}$ a hundred times and make sure the sum is divisible by 3. This can be modeled by the generating function $f(x) = (x^3 + x^5 + x^7 + x^9)^{100}$. We just need sum of the coefficients of f whose exponents are divisible by 3. Let $\omega = e^{\frac{2\pi i}{3}}$. By the roots of unity filter, we need to compute

$$\frac{f(1) + f(\omega) + f(\omega^2)}{3}.$$

Using the fact that $1+\omega+\omega^2=0$ (which can be proved from $\omega^3-1=0$), this will all simplify to

$$\frac{2^{200}+2}{3}.$$

Note by Fermat's Little Theorem, we have

$$2^{28} \equiv 1 \pmod{29}.$$

Hence,

$$2^{200} \equiv 2^4 = 16 \pmod{29}.$$

Thus, our final answer is

$$\frac{16+2}{3} = \boxed{6}.$$

Wednesday (Math Prize for Girls). Compute the number of integers n between 1 and 2019 inclusive such that

$$\prod_{k=0}^{n-1}\left(\left(1+e^{\frac{2\pi ik}{n}}\right)^n+1\right)=0.$$

Solution. Note one of the terms in the product must be 0 for the entire product to be 0. Hence, there exists a z such that

$$\left(1+e^{\frac{2\pi iz}{n}}\right)^n=-1.$$

Taking the magnitude of both sides gives that

$$\left|1+e^{\frac{2\pi iz}{n}}\right|=1.$$

Let $a=e^{\frac{2\pi iz}{n}}$. We know $|a|=1$ and also that $|1+a|=1$. There are only two such values of a. They are the third roots of unity except 1! Hence, $\frac{z}{n}$ is either $\frac{1}{3}$ or $\frac{2}{3}$. That means 3 divides n. Let $n=3b$ for an integer b. Then, z is either b or $2b$. One can verify that

$$1+e^{\frac{2\pi iz}{n}}=e^{\pm\frac{i\pi}{3}}.$$

Then,

$$\left(1+e^{\frac{2\pi iz}{n}}\right)^{3b}=\left(e^{\pm\frac{i\pi}{3}}\right)^{3b}=(e^{i\pi})^b=(-1)^b.$$

We need to get -1, so b must be odd. Hence, the n we need is just all odd multiples of 3 from 1 to 2019. It is not hard to calculate this to be $\boxed{337}$.

Thursday (Titu). For positive integers n, define

$$f(n) = \sum_{k=0}^{n-1} \cos^{2n}\left(\frac{k\pi}{n}\right).$$

Compute

$$\sum_{k=2}^{\infty} \frac{f(k)}{k \cdot 2^k}.$$

Solution. Let $\omega = e^{\frac{2\pi i}{n}}$. Note that

$$\begin{aligned}
f(n) &= \sum_{k=0}^{n-1} \cos^{2n}\left(\frac{k\pi}{n}\right) \\
&= \sum_{k=0}^{n-1} \left(\frac{\cos\left(\frac{2k\pi}{n}\right) + 1}{2}\right)^n \\
&= \frac{1}{4^n} \cdot \sum_{k=0}^{n-1} \left(2\cos\left(\frac{2k\pi}{n}\right) + 2\right)^n \\
&= \frac{1}{4^n} \cdot \sum_{k=0}^{n-1} \left(2 + \omega^k + \frac{1}{\omega^k}\right)^n \\
&= \frac{1}{4^n} \cdot \sum_{k=0}^{n-1} \left(\frac{2\omega^k + \omega^{2k} + 1}{\omega^k}\right)^n \\
&= \frac{1}{4^n} \cdot \sum_{k=0}^{n-1} \left(1 + \omega^k\right)^{2n}.
\end{aligned}$$

By roots of unity filter on $(1+x)^{2n}$, we know that

$$\sum_{k=0}^{n-1} (1+\omega^k)^{2n} = n \cdot \left(\binom{2n}{0} + \binom{2n}{n} + \binom{2n}{2n}\right) = n \cdot \left(2 + \binom{2n}{n}\right).$$

Therefore,

$$f(n) = \frac{n}{4^n} \cdot \left(2 + \binom{2n}{n}\right).$$

From here, we have

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{f(k)}{k \cdot 2^k} &= \sum_{k=2}^{\infty} \frac{\frac{k}{4^k} \cdot \left(2 + \binom{2k}{k}\right)}{k \cdot 2^k} \\
&= \sum_{k=2}^{\infty} \frac{2}{8^k} + \sum_{k=0}^{\infty} \left[\frac{\binom{2k}{k}}{8^k}\right] - 1 - \frac{1}{4} \\
&= \boxed{\sqrt{2} - \frac{17}{14}}.
\end{aligned}$$

Friday (Titu). There exists a fraction a such that

$$\left|\frac{6z - i}{2 + 3iz}\right| \leq 1$$

if and only if $|z| \leq a$. Given that a can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers, compute $p + q$.

Solution. Let $z = x + yi$. Then

$$\left|\frac{6z - i}{2 + 3iz}\right| \leq 1$$

becomes

$$|6(x + yi) - i| \leq |2 + 3i(x + yi)| \iff 6x^2 + (6y - 1)^2 \leq (2 - 3y)^2 + (3x)^2.$$

This in turn yields

$$x^2 + y^2 \leq \frac{1}{9} \iff |z| \leq \frac{1}{3},$$

hence $a = \boxed{\frac{1}{3}}$.
